Basic Structures: Sequences and Summations

CS261 Mathematical Foundations of CS
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Sequences and Summations

Section 2.4

Section Summary[4]

Sequences.

- Examples: Geometric Progression, Arithmetic Progression.

Recurrence Relations.

- Example: Fibonacci Sequence.

Summations.

Introduction₂

Sequences are ordered lists of elements.

- 1, 2, 3, 5, 8.
- 1, 3, 9, 27, 81, …….

Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.

We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Sequences [1]

Definition: A *sequence* is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, …..} or {1, 2, 3, 4, ….}) to a set *S*.

The notation a_n is used to denote the image of the integer *n*. We can think of a_n as the equivalent of *f(n)* where *f* is a function from $\{0,1,2,\ldots..\}$ to *S*. We call a_n a *term* of the sequence.

Sequences[2]

Example: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1, a_2, a_3 \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$$

Arithmetic Progression

Definition: A arithmetic progression is a sequence of the form: $a, a+d, a+2d, \ldots, a+nd, \ldots$

where the *initial term a* and the *common difference d* are real numbers.

Examples:

1. Let $a = -1$ and $d = 4$:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

2. Let $a = 7$ and $d = -3$:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let $a = 1$ and $d = 2$:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Geometric Progression

Definition: A *geometric progression* is a sequence of the form: $a, ar^2, \ldots, ar^n, \ldots$

where the *initial term a* and the *common ratio r* are real numbers.

Examples:

1. Let $a = 1$ and $r = -1$. Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

2. Let $a = 2$ and $r = 5$. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

3. Let $a = 6$ and $r = 1/3$. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \left\{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\right\}$$

Strings [1]

Definition: A *string* is a finite sequence of characters from a finite set (an alphabet).

Sequences of characters or bits are important in computer science.

The *empty string* is represented by λ.

The string *abcde* has *length* 5.

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Questions about Recurrence Relations [1]

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 1,2,3,4,.... and suppose that $a_0 = 2$. What are a_1, a_2 and a_3?

[Here $a_0 = 2$ is the initial condition.]

Solution: We see from the recurrence relation that

$$a_1 = a_0 + 3 = 2 + 3 = 5$$

$$a_2 = 5 + 3 = 8$$

$$a_3 = 8 + 3 = 11$$

Questions about Recurrence Relations 2

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for n = 2,3,4,.... and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

Solution: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots,$ by:

- Initial Conditions: $f_0 = 0,\ f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6.

Answer :

$f_2 = f_1 + f_0 = 1 + 0 = 1,$

$f_3 = f_2 + f_1 = 1 + 1 = 2,$

$f_4 = f_3 + f_2 = 2 + 1 = 3,$

$f_5 = f_4 + f_3 = 3 + 2 = 5,$

$f_6 = f_5 + f_4 = 5 + 3 = 8.$

Solving Recurrence Relations

Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.

Such a formula is called a *closed formula*.

Various methods for solving recurrence relations will be covered in Chapter 8 where recurrence relations will be studied in greater depth.

Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of induction (Chapter 5).

Iterative Solution Example[1]

Method 1: Working upward, forward substitution Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 2,3,4,.... and suppose that $a_1 = 2$.

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

$$\vdots$$

$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n-1)$$

Iterative Solution Example 2

Method 2: Working downward, backward substitution Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 2,3,4,.... and suppose that $a_1 = 2$.

$$a_n = a_{n-1} + 3$$
$$= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2$$
$$= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3$$
$$\vdots$$
$$= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1)$$

Questions on Special Integer Sequences (*opt*) 2

TABLE 1 Some Useful Sequences.

nth Term	***First 10 Terms***
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100,...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000,...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000,...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024,...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049,...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800,...

Summations[1]

Sum of the terms $a_m, a_m+1, \ldots, a_n$

from the sequence $\{a_n\}$

The notation:

$$\sum_{j=m}^{n} a_j \quad \sum\nolimits_{j=m}^{n} a_j \quad \sum\nolimits_{m \le j \le n} a_j$$

represents

$$a_m + a_{m+1} + \cdots + a_n$$

The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit m* and ending with its *upper limit n*.

Summations 2

More generally for a set S:

$$\sum_{j \in s} a_j$$

Examples:

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_0^n r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_1^\infty \frac{1}{i}$$

If $S = \{2,5,7,10\}$ then $\sum_{j \in s} a_j = a_2 + a_5 + a_7 + a_{10}$

Geometric Series[1]

Sums of terms of geometric progressions

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

Proof: Let $S_n = \sum_{j=0}^{n} ar^j$

To compute S_n, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$rS_n = r\sum_{j=0}^{n} ar^j$$

$$= \sum_{j=0}^{n} ar^{j+1}$$

Geometric Series 2

$$= \sum_{j=0}^{n} ar^{j+1}$$ From previous slide.

$$= \sum_{k=1}^{n+1} ar^{k}$$ Shifting the index of summation with $k = j + 1$.

$$= \left(\sum_{k=0}^{n} ar^{k} \right) + \left(ar^{n+1} - a \right)$$ Removing $k = n + 1$ term and adding $k = 0$ term.

$$= S_n + \left(ar^{n+1} - a \right)$$ Substituting S for summation formula

$$\therefore \quad rS_n = S_n + \left(ar^{n+1} - a \right)$$

$$S_n = \frac{ar^{n+1} - a}{r - 1}$$ if $r \neq 1$

$$S_n = \sum_{j=0}^{n} ar^{j} = \sum_{j=0}^{n} a = (n+1)a$$ if $r = 1$

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	***Closed From***
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\dfrac{ar^{n+1}-a}{r-1},\ r \neq 1$
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\dfrac{1}{1-x}$
$\sum_{k=0}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\dfrac{1}{(1-x)^2}$

Geometric Series: We just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus)

Access the text alternative for slide images.