Basic Structures: Sets and Functions

CS261 Mathematical Foundations of CS
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Spring 2024
University of New Mexico

Set Operations

Section 2.2

Section Summary 2

Set Operations.

- Union.
- Intersection.
- Complementation.
- Difference.

More on Set Cardinality.

Set Identities.

Proving Identities.

Membership Tables.

Boolean Algebra

Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra*. This is discussed in Chapter 12.

The operators in set theory are analogous to the corresponding operator in propositional calculus.

As always there must be a universal set U. All sets are assumed to be subsets of U.

Union

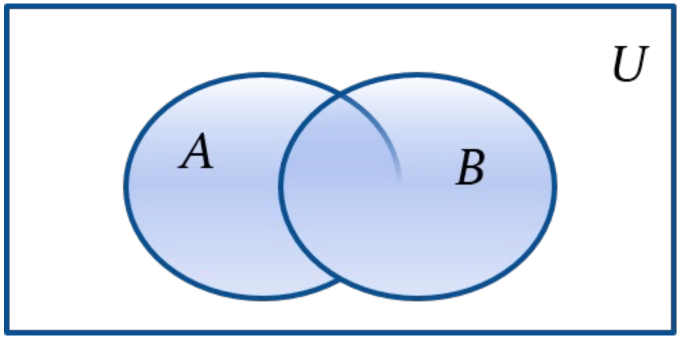
Definition: Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set:

$$\{x \mid x \in A \vee x \in B\}$$

Example: What is $\{1,2,3\} \cup \{3,\ 4,\ 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



Intersection

Definition: The *intersection* of sets *A* and *B*, denoted by $A \cap B$, is

$$\{x \mid x \in A \wedge x \in B\}$$

Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.

Example: What is? $\{1,2,3\} \cap \{3,4,5\}$?

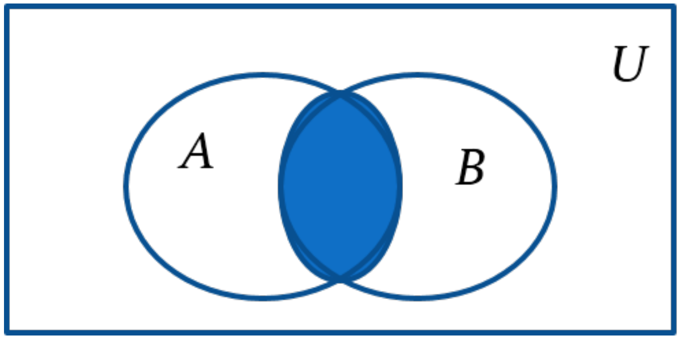
Solution: $\{3\}$

Example: What is?

$\{1,2,3\} \cap \{4,5,6\}$?

Solution: ∅

Venn Diagram for A ∩B



Complement

Definition: If A is a set, then the *complement* of the A (with respect to U), denoted by $\bar{A}$ is the set U - A

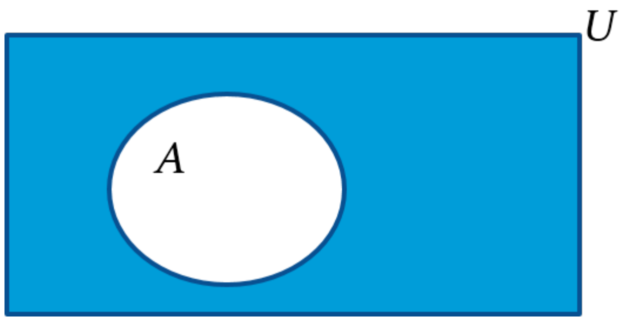
$$\bar{A} = \{x \mid x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

Venn Diagram for Complement

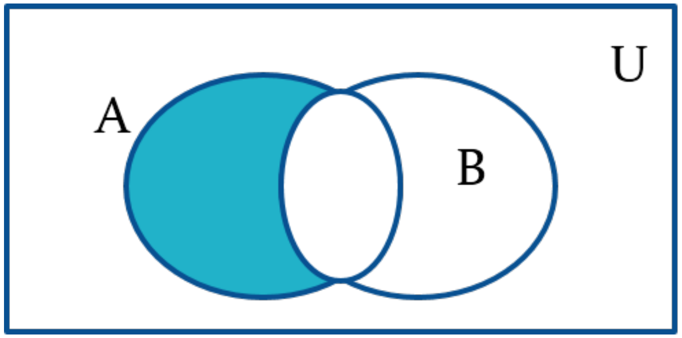


Difference

Definition: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \land x \notin B\} = A \cap \overline{B}$$

Venn Diagram for $A - B$



The Cardinality of the Union of Two Sets

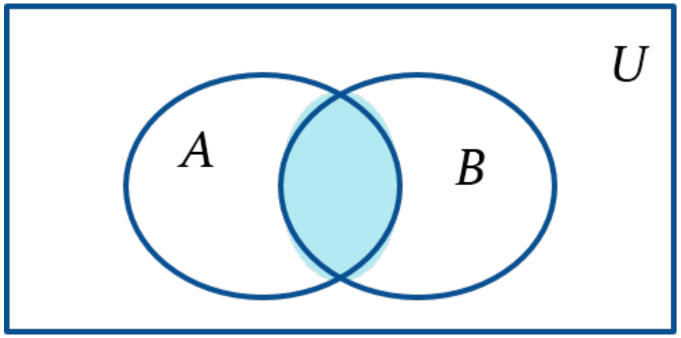
Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: Let *A* be the math majors in your class and *B* be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of *n* sets, where *n* is a positive integer.

Venn Diagram for A, B, A ∩ B, A ∪ B



Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$

2. $A \cap B$

   Solution: $\{4,5\}$

3. $\overline{A}$

   Solution: $\{0,6,7,8,9,10\}$

4. $\overline{B}$

   Solution: $\{0,1,2,3,9,10\}$

5. $A - B$

   Solution: $\{1,2,3\}$

6. $B - A$

   Solution: $\{6,7,8\}$

Symmetric Difference

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

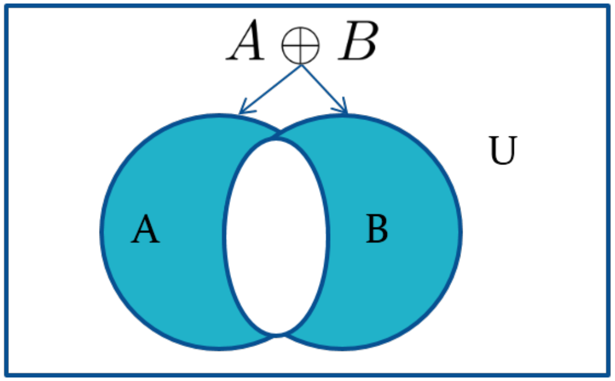
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is $A \oplus B$:

Solution: $\{1,2,3,6,7,8\}$



Venn Diagram

Set Identities [1]

Identity laws

$$A \cup \varnothing = A \qquad A \cap U = A$$

Domination laws

$$A \cup U = U \qquad A \cap \varnothing = \varnothing$$

Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

Complementation law

$$\left(\overline{\overline{A}}\right) = A$$

Set Identities 2

Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities[3]

De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$$

Proving Set Identities

Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other.
2. Use set builder notation and propositional logic.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proof of Second De Morgan Law [1]

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

$1)\ \overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

$2)\ \overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Proof of Second De Morgan Law 2

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg\left(\left(x \in A\right) \wedge \left(x \in B\right)\right)$	by defn. of intersection
$\neg\left(x \in A\right) \vee \neg\left(x \in B\right)$	1st De Morgan law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	by defn. of union

Proof of Second De Morgan Law 3

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$\left(x \in \overline{A}\right) \vee \left(x \in \overline{B}\right)$	by defn. of union
$\left(x \notin A\right) \vee \left(x \in \overline{B}\right)$	defn. of complement
$\neg\left(x \in A\right) \vee \neg\left(x \in B\right)$	defn. of negation
$\neg\left(\left(x \in A\right) \wedge \neg\left(x \in B\right)\right)$	1st De Morgan law for Prop Logic
$\neg\left(x \in A \cap B\right)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Set-Builder Notation: Second De Morgan Law

$$\overline{A \cap B} = x \in \overline{A \cap B}$$ by defn. of complement

$$= \{x \mid \neg(x \in (A \cap B))\}$$ by defn. of does not belong symbol

$$= \{x \mid \neg(x \in A \wedge x \in B\}$$ by defn. of intersection

$$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$$ by 1st De Morgan law for Prop Logic

$$= \{x \mid x \notin A \vee x \notin B\}$$ by defn. of not belong symbol

$$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$$ by defn. of complement

$$= \{x \mid x \in \overline{A} \cup \overline{B}\}$$ by defn. of union

$$= \overline{A} \cup \overline{B}$$ by meaning of notation

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Functions

Section 2.3

Section Summary 3

Definition of a Function.

- Domain, Codomain.
- Image, Preimage.

Injection, Surjection, Bijection.

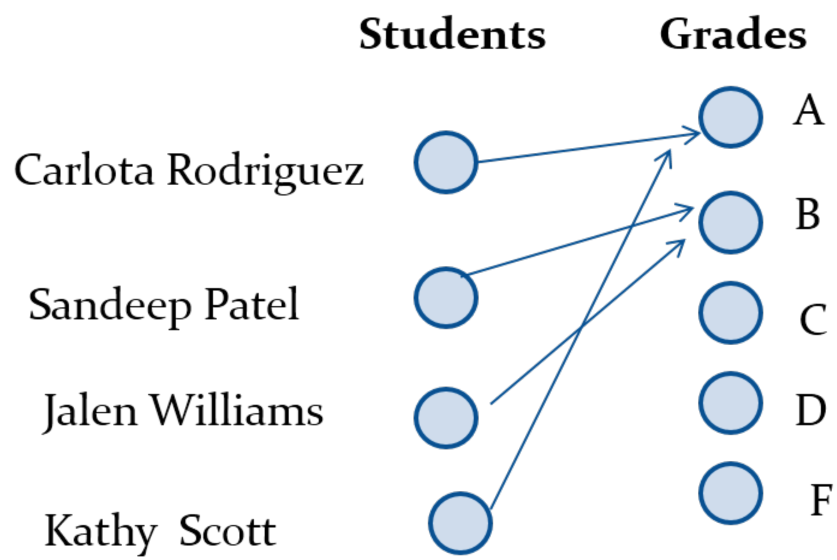
Inverse Function.

Function Composition.

Functions[1]

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \rightarrow B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Functions$_2$

A function f: $A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.

Specifically, a function *f* from *A* to *B* contains one, and only one ordered pair (*a, b*) for every element $a \in A$.

$$\forall x \left[x \in A \rightarrow \exists y \left[y \in B \wedge (x, y) \in f \right] \right]$$

and

$$\forall x, y_1, y_2 \left[\left[(x, y_1) \in f \wedge (x, y_2) \in f \right] \rightarrow y_1 = y_2 \right]$$

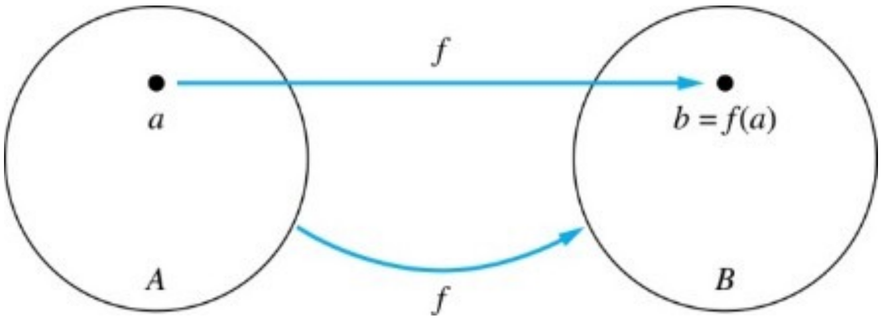
Functions[3]

Given a function f: $A \rightarrow B$**:**

We say f *maps* A to B *or* f is a *mapping* from A to B.

A is called the *domain* of f.

B is called the *codomain* of f.



If $f(a) = b$,

- then b is called the *image* of a under f.
- a is called the *preimage* of b.

The range of f is the set of all images of points in **A** under f. We denote it by $f(\boldsymbol{A})$.

Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

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Representing Functions

Functions may be specified in different ways:

An explicit statement of the assignment. Students and grades example.

A formula.

$$f(x) = x + 1$$

A computer program.

- A Java program that when given an integer *n*, produces the *n*th Fibonacci Number (covered in the next section and also in Chapter 5).

Questions

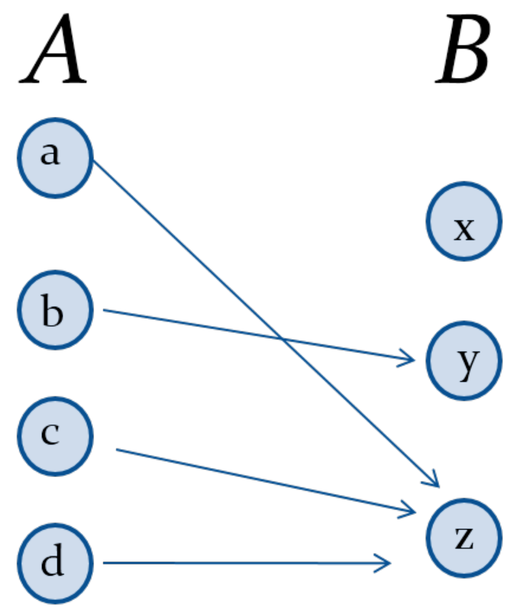
$f(a)$ = ? z

The image of d is ? z

The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b



$f(A)$ = ? $\{y,z\}$

The preimage(s) of z is (are) ? $\{a,c,d\}$

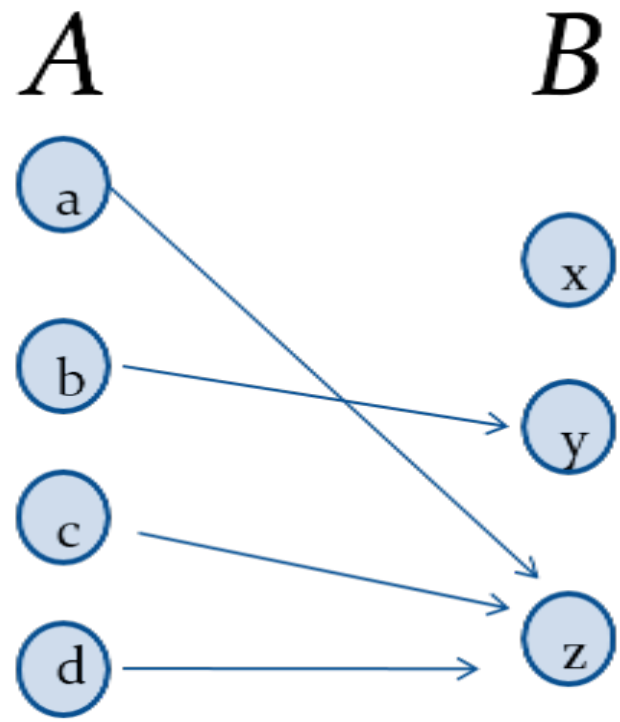
Question on Functions and Sets

If $f: A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) \mid s \in S\}$$

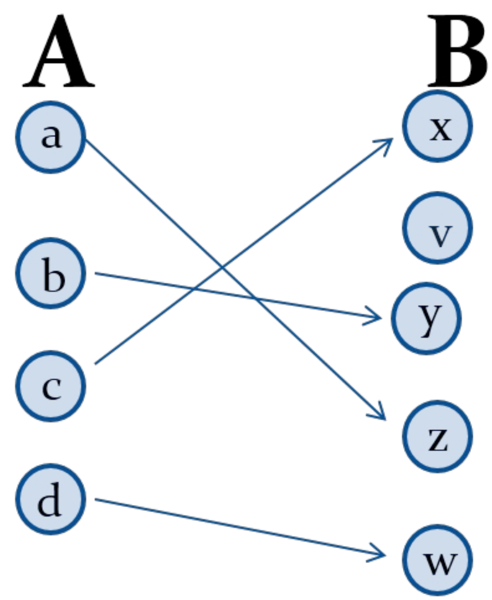
$f\{a,b,c,\}$ is ? $\{y,z\}$

$f\{c,d\}$ is ? $\{z\}$



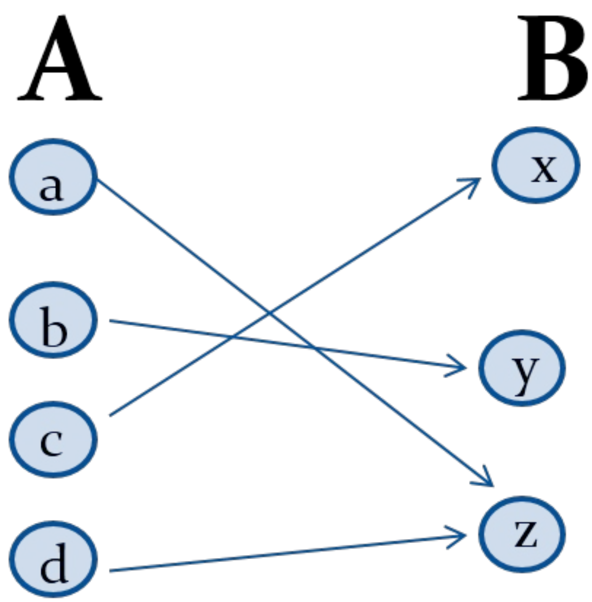
One-to-one

Definition: A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.



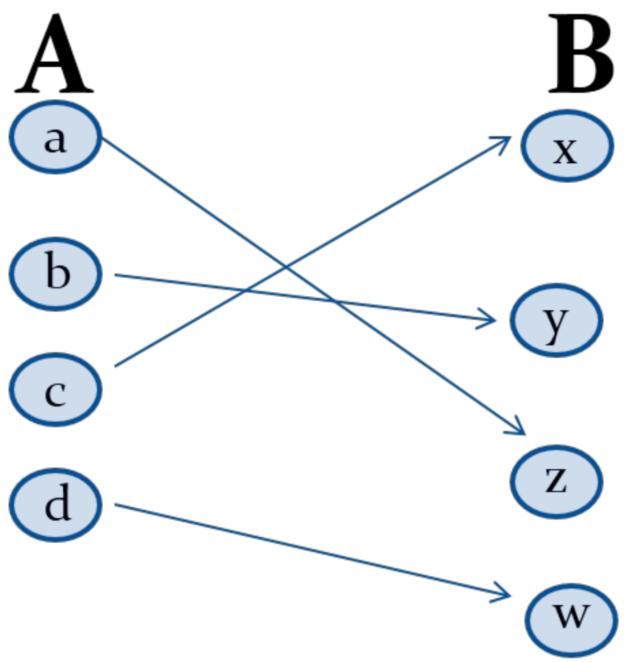
Onto

Definition: A function *f* from *A* to *B* is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
A function *f* is called a *surjection* if it is *onto*.



Invertible

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



Showing that f is one-to-one or onto [1]

Suppose that $f : A \rightarrow B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Showing that *f* is one-to-one or onto 2

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

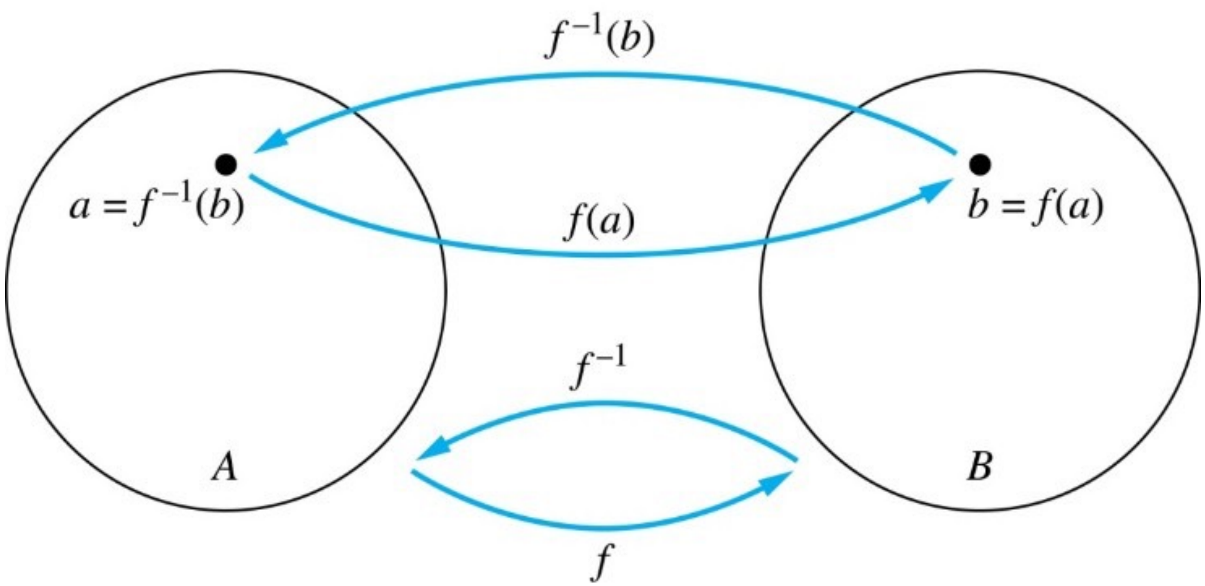
Example 2: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

Inverse Functions [1]

Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from *B* to *A* defined as $f^{-1}(y) = x$ iff $f(x) = y$

No inverse exists unless *f* is a bijection. Why?



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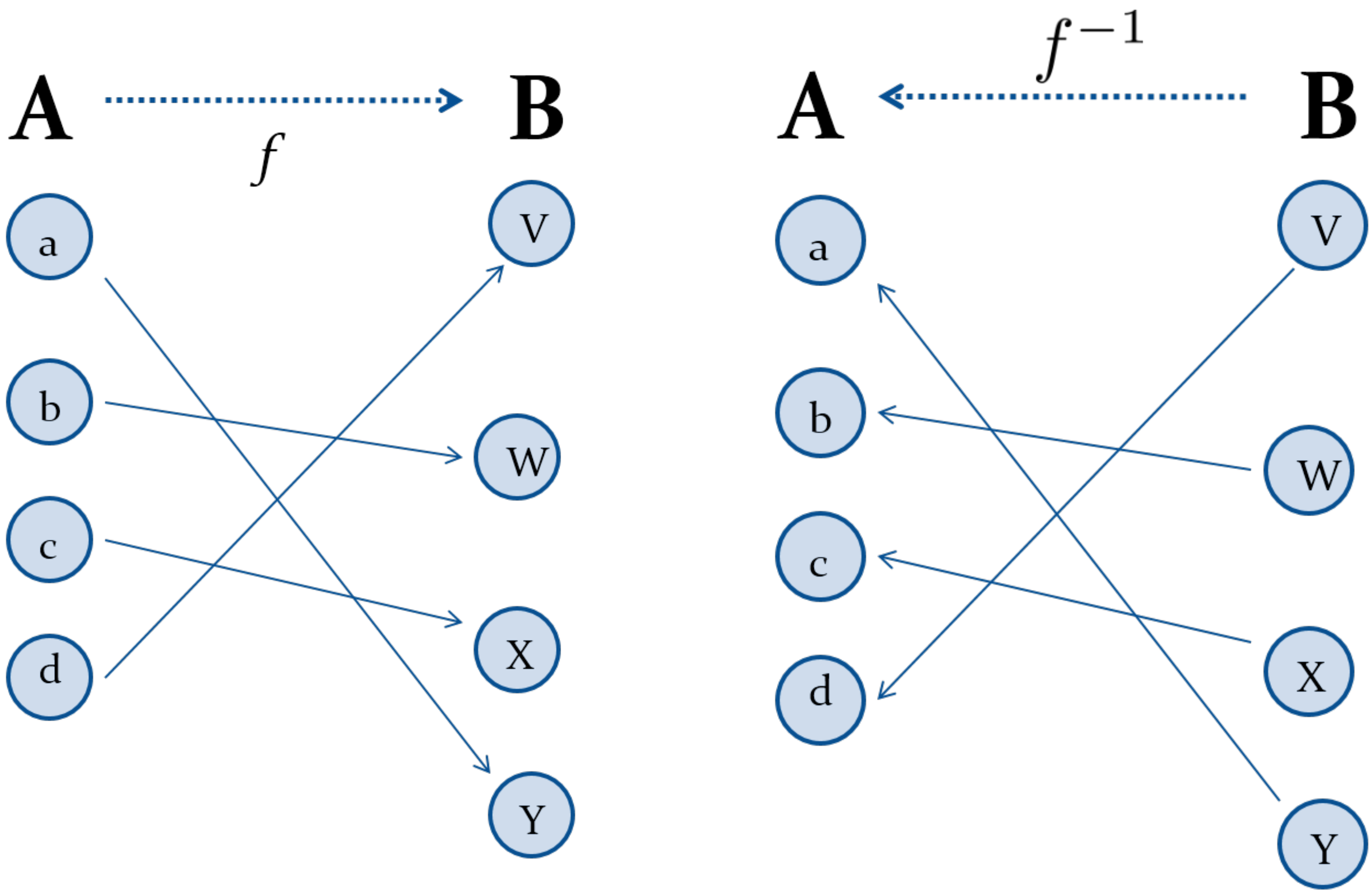
Inverse Functions 2



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Questions [1]

Example 1: Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions 2

Example 2: Let f: **Z** → **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Questions 3

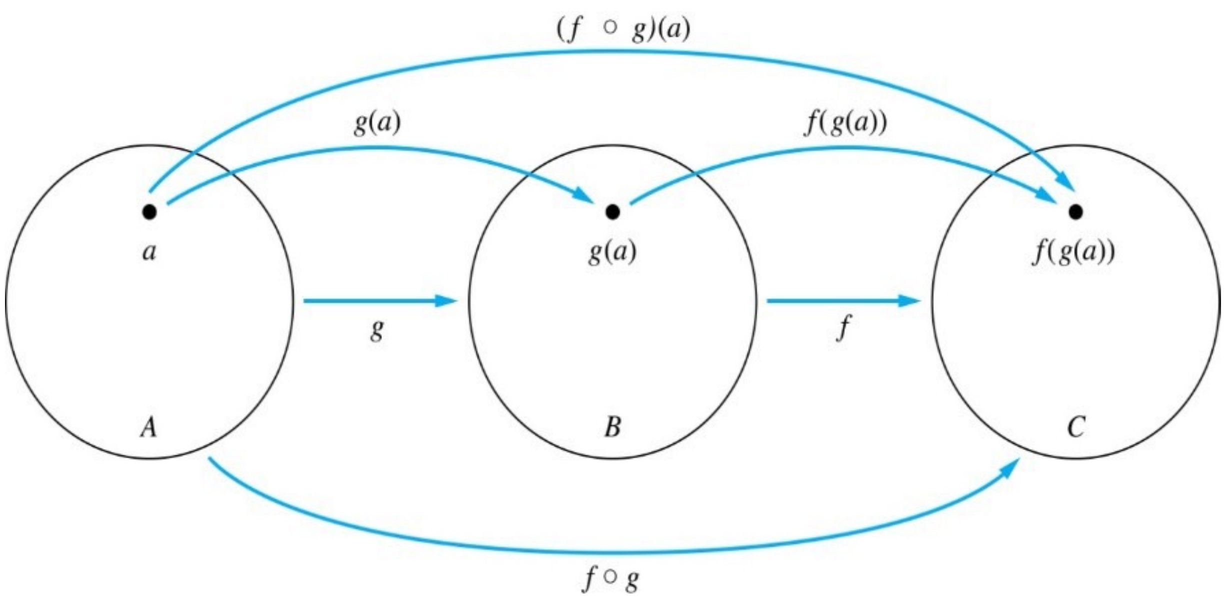
Example 3: Let *f*: **R** → **R** be such that $f(x) = x^2$
Is *f* invertible, and if so, what is its inverse?

Solution: The function *f* is not invertible because it is not one-to-one.

Composition[1]

Definition: Let f: $B \rightarrow C$, g: $A \rightarrow B$. The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by

$$f \circ g(x) = f(g(x))$$



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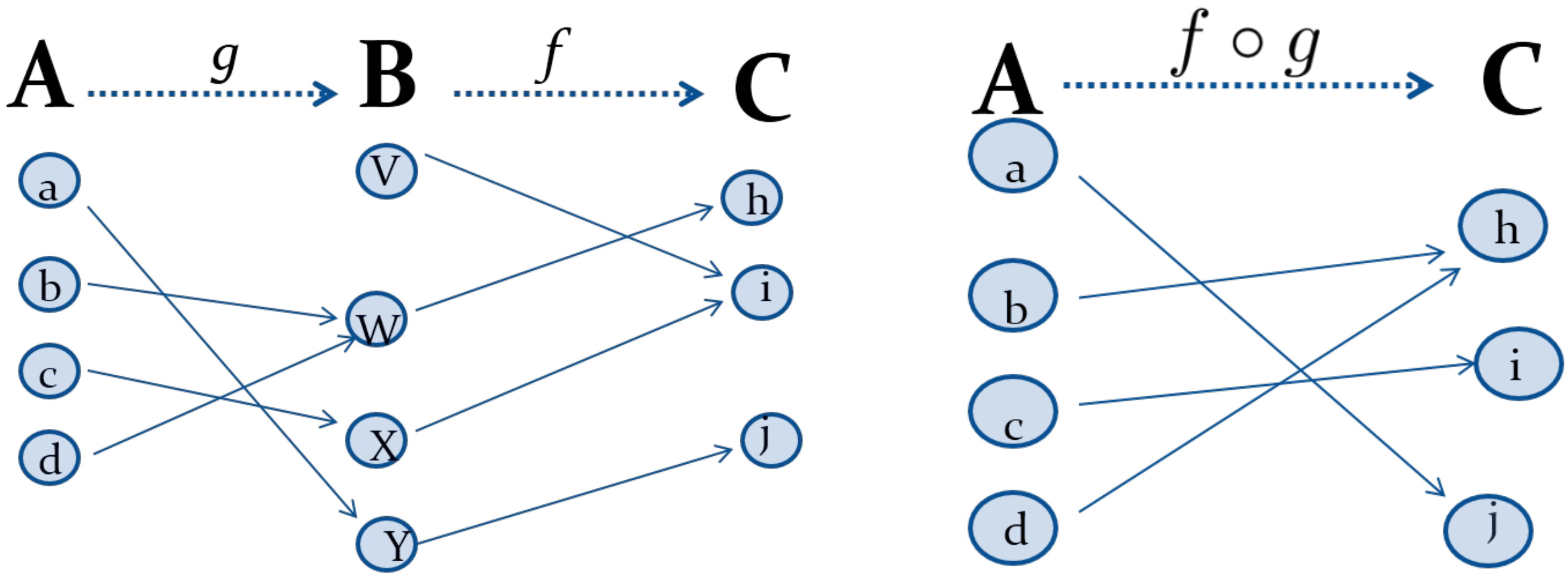
Composition 2



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Composition 3

Example 1: If

$$f(x) = x^2 \text{ and } g(x) = 2x + 1,$$

then

$$f(g(x)) = (2x + 1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions 1

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

Solution: The composition $f \circ g$ is defined by

$$f \circ g(a) = f(g(a)) = f(b) = 2.$$
$$f \circ g(b) = f(g(b)) = f(c) = 1.$$
$$f \circ g(c) = f(g(c)) = f(a) = 3.$$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions 2

Example 2: Let f and g be functions from the set of integers to the set of integers defined by

$$f(x) = 2x + 3 \quad \text{and} \quad g(x) = 3x + 2.$$

What is the composition of *f* and *g*, and also the composition of *g* and *f* ?

Solution:

$$f \circ g(x) = f(g(x)) = f(3x+2) = 2(3x+2) + 3 = 6x + 7$$

$$g \circ f(x) = g(f(x)) = g(3x+2) = 3(3x+2) + 2 = 6x + 11$$