Basic Structures: Sets

CS261 Mathematical Foundations of CS

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Chapter Summary

Sets

- The Language of Sets.
- Set Operations.
- Set Identities.

Functions

- Types of Functions.
- Operations on Functions.
- Computability.

Sequences and Summations

- Types of Sequences.
- Summation Formulae.

Set Cardinality

- Countable Sets.

Matrices

- Matrix Arithmetic.

Sets

Section 2.1

Section Summary 1

Definition of sets.

Describing Sets.

- Roster Method.
- Set-Builder Notation.

Some Important Sets in Mathematics.

Empty Set and Universal Set.

Subsets and Set Equality.

Cardinality of Sets.

Tuples.

Cartesian Product.

Introduction [1]

Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.

- Important for counting.
- Programming languages have set operations.

Set theory is an important branch of mathematics.

- Many different systems of axioms have been used to develop set theory.
- Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

A *set* is an unordered collection of objects.

- the students in this class.
- the chairs in this room.

The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.

The notation $a \in A$ denotes that *a* is an element of the set *A*.

If *a* is not a member of *A*, write $a \notin A$.

Describing a Set: Roster Method

$$S = \{a, b, c, d\}$$

Order not important.

$$S = \{a, b, c, d\} = \{b, c, a, d\}$$

Each distinct object is either a member or not; listing more than once does not change the set.

$$S = \{a, b, c, d\} = \{a, b, c, b, c, d\}$$

Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

$$S = \{a, b, c, d, \ \ldots\ldots, z\}$$

Roster Method

Set of all vowels in the English alphabet:

$$V = \{a, e, i, o, u\}$$

Set of all odd positive integers less than 10:

$$O = \{1, 3, 5, 7, 9\}$$

Set of all positive integers less than 100:

$$S = \{1, 2, 3, \ldots\ldots, 99\}$$

Set of all integers less than 0:

$$S = \{\ldots, -3, -2, -1\}$$

Some Important Sets

N = *natural numbers* = $\{0,1,2,3\ldots.\}$

Z = *integers* = $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$

$\mathbf{Z}^+$ = *positive integers* = $\{1,2,3,\ldots..\}$

R = set of *real numbers.*

$\mathbf{R}^+$ = set of *positive real numbers.*

C = set of *complex numbers*.

Q = set of rational numbers.

Set-Builder Notation

Specify the property or properties that all members must satisfy:

$$S = \{x \mid x \text{ is a positive integer less than } 100\}$$

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$

$$O = \{x \in \mathbf{Z}^{+} \mid x \text{ is odd and } x < 10\}$$

A predicate may be used:

$$S = \{x \mid P(x)\}$$

Example: $S = \{x \mid \text{Prime}(x)\}$

Positive rational numbers:

$$\mathbf{Q}^{+} = \{x \in \mathbf{R} \mid x = p/q, \text{for some positive integers } p,q\}$$

Interval Notation

$$[a,b] = \{x \mid a \le x \le b\}$$
$$[a,b) = \{x \mid a \le x < b\}$$
$$(a,b] = \{x \mid a < x \le b\}$$
$$(a,b) = \{x \mid a < x < b\}$$

closed interval $[\mathrm{a,b}]$

open interval $(\mathrm{a,b})$

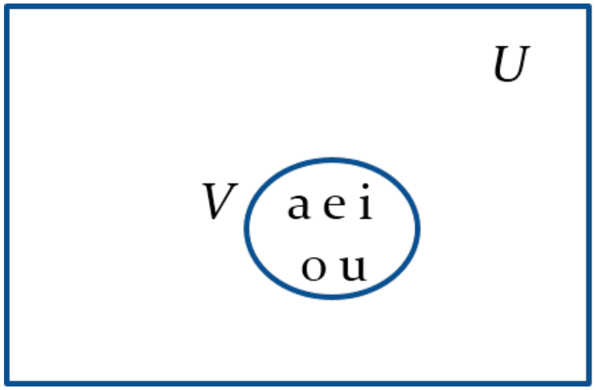
Universal Set and Empty Set

The *universal set U* is the set containing everything currently under consideration.

- Sometimes implicit.
- Sometimes explicitly stated.
- Contents depend on the context.

The empty set is the set with no elements. Symbolized ∅, but {} also used.

Venn Diagram



John Venn (1834-1923)
Cambridge, UK

Russell's Paradox

Let *S* be the set of all sets which are not members of themselves. A paradox results from trying to answer the question "Is *S* a member of itself?"

Related Paradox:

- Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question "Does Henry shave himself?"

Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner

Some things to remember

Sets can be elements of sets.

$$\{\{1,2,3\}, a, \{b,c\}\}$$

$$\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$$

The empty set is different from a set containing the empty set.

$$\emptyset \neq \{\emptyset\}$$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x \left(x \in A \leftrightarrow x \in B \right)$
- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3,\ 5,\ 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x \left(x \in A \rightarrow x \in B \right)$ is true.
    1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$, for every set S.
    2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set S.

Showing a Set is or is not a Subset of Another Set

Showing that A is a Subset of B: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.

Showing that A is not a Subset of B: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

Recall that two sets *A* and *B* are *equal*, denoted by *A* = *B*, iff

$$\forall x\left(x \in A \leftrightarrow x \in B\right)$$

Using logical equivalences we have that *A* = *B* iff

$$\forall x\left[\left(x \in A \rightarrow x \in B\right) \wedge \left(x \in B \rightarrow x \in A\right)\right]$$

This is equivalent to

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

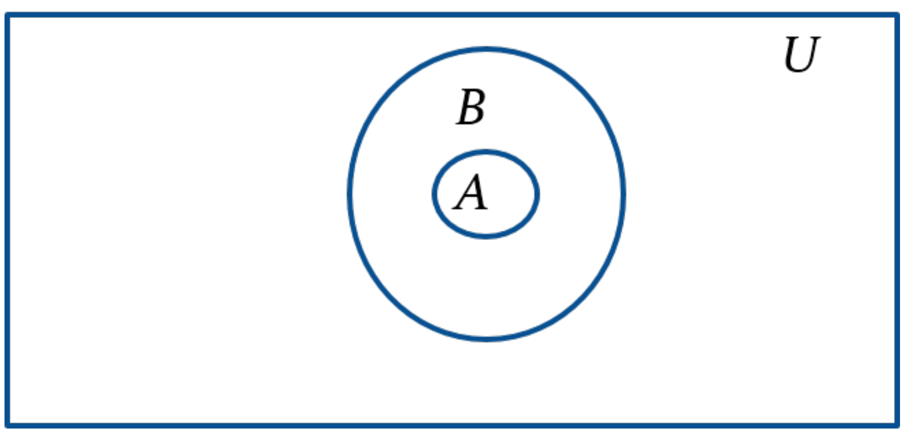
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x \quad (x \in A \rightarrow x \in B) \wedge \exists x (x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Set Cardinality

Definition: If there are exactly n distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set *A*, denoted by $|A|$, is the number of (distinct) elements of *A*.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set *A*, denoted *P*(*A*), is called the *power set* of *A*.

Example: If $A = \{a, b\}$ then

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

If a set has *n* elements, then the cardinality of the power set is 2^n. (In Chapters 5 and 6, we will discuss different ways to show this.)

Tuples

The *ordered n-tuple* $(a_1, a_2, \ldots\ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.

Two n-tuples are equal if and only if their corresponding elements are equal.

2-tuples are called *ordered pairs*.

The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product[1]

René Descartes
(1596-1650)

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

Example:

$$A = \{a,b\} \quad B = \{1,2,3\}$$

$$A \times B = \{(a,1),(a,2),(a,3),\ (b,1),(b,2),(b,3)\}$$

Definition: A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

Cartesian Product 2

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered *n*-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2 \cdots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}, B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C$ **=** {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)}

Truth Sets of Quantifiers

Given a predicate *P* and a domain *D*, we define the *truth set* of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

$$\{x \in D \mid P(x)\}$$

Example: The truth set of *P*(*x*) where the domain is the integers and *P*(*x*) is "$|x| = 1$" is the set $\{-1,1\}$