Predicate Logic

CS261 Mathematical Foundations of CS
Professor Leah Buechley
Spring 2024
University of New Mexico

Summary

Predicate Logic (First-Order Logic (FOL), Predicate Calculus).

- The Language of Quantifiers.
- Logical Equivalences.
- Nested Quantifiers.
- Translation from Predicate Logic to English.
- Translation from English to Predicate Logic.

Predicates and Quantifiers

Section 1.4

Section Summary 1

Predicates.

Variables.

Quantifiers.

- Universal Quantifier.
- Existential Quantifier.

Negating Quantifiers.

- De Morgan's Laws for Quantifiers.

Translating English to Logic.

Propositional Logic Not Enough

If we have:

"All men are mortal."

"Socrates is a man."

Does it follow that "Socrates is mortal?"

Can't be represented in propositional logic. Need a language that talks about objects, their properties, and their relations.

Later we'll see how to draw inferences.

Introducing Predicate Logic

Predicate logic uses the following new features:

- Variables: *x*, *y*, *z.*
- Predicates: $P(x)$, $M(x)$.
- Quantifiers (*to be covered in a few slides*):

Propositional functions are a generalization of propositions.

- They contain variables and a predicate, e.g., $P(x)$.
- Variables can be replaced by elements from their *domain*.

Propositional Functions

Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or *bound* by a quantifier, as we will see later).

The statement *P(x)* is said to be the value of the propositional function *P* at *x*.

For example, let *P(x)* denote "$x > 0$" and the domain be the integers. Then:

P(-3) is false.

P(0) is false.

P(3) is true.

Often the domain is denoted by *U*. So in this example *U* is the integers.

Examples of Propositional Functions

Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

R(2,-1,5)

Solution: F

R(3,4,7)

Solution: T

R(x, 3, z)

Solution: Not a Proposition

Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

Q(2,-1,3)

Solution: T

Q(3,4,7)

Solution: F

Q(x, 3, z)

Solution: Not a Proposition

Compound Expressions

Connectives from propositional logic carry over to predicate logic.

If *P(x)* denotes "$x > 0$," find these truth values:

$P(3) \lor P(-1)$ **Solution**: T

$P(3) \land P(-1)$ **Solution**: F

$P(3) \rightarrow P(-1)$ **Solution**: F

$P(3) \rightarrow \neg P(-1)$ **Solution**: T

Expressions with variables are not propositions and therefore do not have truth values. For example,

$P(3) \land P(y)$

$P(x) \rightarrow P(y)$

When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers

Charles
Peirce
(1839-1914)

We need *quantifiers* to express the meaning of English words including *all* and *some*:

- "All men are Mortal."
- "Some cats do not have fur."

The two most important quantifiers are:

- *Universal Quantifier,* **"**For all," symbol: $\forall$
- *Existential Quantifier*, "There exists," symbol: $\exists$

We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.

$\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.

$\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.

The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

$\forall x\, P(x)$ is read as "For all x, $P(x)$" or "For every x, $P(x)$"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.

2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.

3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall x\, P(x)$ is false.

Existential Quantifier

$\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

Thinking about Quantifiers

When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.

To evaluate $\forall x\, P(x)$ loop through all x in the domain.

- If at every step P(x) is true, then $\forall x\, P(x)$ is true.
- If at a step P(x) is false, then $\forall x\, P(x)$ is false and the loop terminates.

To evaluate $\exists x\, P(x)$ loop through all x in the domain.

- If at some step, P(x) is true, then $\exists x\, P(x)$ is true and the loop terminates.
- If the loop ends without finding an x for which P(x) is true, then $\exists x\, P(x)$ is false.

Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Thinking about Quantifiers as Conjunctions and Disjunctions

If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.

If U consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Properties of Quantifiers

The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.

Examples:

1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.

2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.

3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

For example, $\forall x\, P(x) \vee Q(x)$ means $\left(\forall x\, P(x)\right) \vee Q(x)$

$\forall x\, \left(P(x) \vee Q(x)\right)$ means something different.

Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, \left(P(x) \vee Q(x)\right)$.

Translating from English to Logic[1]

Example 1: Translate the following sentence into predicate logic: “Every student in this class has taken a course in Java.”

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, define a propositional function J(*x*) denoting “x has taken a course in Java” and translate as $\forall x\, J(x)$.

Solution 2: But if *U* is all people, also define a propositional function S(x) denoting “x is a student in this class” and translate as $\forall x\big(S(x) \rightarrow J(x)\big)$.

$\forall x\ \big(S(x) \wedge J(x)\big)$ is not correct. What does it mean?

Translating from English to Logic 2

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 2: But if *U* is all people, then translate as

$$\exists x\, \big(S(x) \wedge J(x)\big)$$

$\exists x\big(S(x) \rightarrow J(x)\big)$ is not correct. What does it mean?

Returning to the Socrates Example

Introduce the propositional functions *Man(x)* denoting "*x* is a man" and *Mortal(x)* denoting "*x* is mortal." Specify the domain as all people.

The two premises are: $\forall x\left(Man(x) \rightarrow Mortal(x)\right)$

$$Man(Socrates)$$

The conclusion is: $Mortal(Socrates)$

Later we will show how to prove that the conclusion follows from the premises.

Equivalences in Predicate Logic

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value

- for every predicate substituted into these statements and
- for every domain of discourse used for the variables in the expressions.

The notation $S \equiv T$ indicates that S and T are logically equivalent.

Example: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Negating Quantified Expressions [1]

Consider $\forall x\, J(x)$

"Every student in your class has taken a course in Java."

Here $J(x)$ is "x has taken a course in Java" and

the domain is students in your class.

Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

Symbolically $\neg \forall x\, J(x)$ and $\exists x \neg J(x)$ are equivalent.

Negating Quantified Expressions 2

Now Consider $\exists x\, J(x)$

"There is a student in this class who has taken a course in Java."

Where $J(x)$ is "x has taken a course in Java."

Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent.

De Morgan's Laws for Quantifiers

The rules for negating quantifiers are:

TABLE 2 De Morgan's Laws for Quantifiers.			
Negation	***Equivalent Statement***	***When Is Negation True?***	***When False?***
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

These are important. You will use these.

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

$$\exists x\left(S(x) \wedge M(x)\right)$$

1. "Every student in this class has visited Canada or Mexico."

   Solution: Add $C(x)$ denoting "x has visited Canada."

$$\forall x\left(S(x) \rightarrow \left(M(x) \vee C(x)\right)\right)$$

Some Fun with Translating from English into Logical Expressions 1

U = {fleegles, snurds, thingamabobs}

F(x): *x* is a fleegle

S(x): *x* is a snurd

T(x): x is a thingamabob

Translate "Everything is a fleegle"

Solution: $\forall x\, F(x)$

Some Fun with Translating from English into Logical Expressions 2

U = {fleegles, snurds, thingamabobs}

$F(x)$: x is a fleegle

$S(x)$: x is a snurd

$T(x)$: x is a thingamabob

"Nothing is a snurd."

Solution: $\neg\exists x\, S(x)$ What is this equivalent to?

Solution: $\forall x \neg S(x)$

Some Fun with Translating from English into Logical Expressions 3

U = {fleegles, snurds, thingamabobs}

F(x): *x* is a fleegle

S(x): *x* is a snurd

T(x): *x* is a thingamabob

"All fleegles are snurds."

Solution: $\forall x\left(F(x) \rightarrow S(x)\right)$

Some Fun with Translating from English into Logical Expressions 4

U = {fleegles, snurds, thingamabobs}

F(x): *x* is a fleegle

S(x): *x* is a snurd

T(x): *x* is a thingamabob

"Some fleegles are thingamabobs."

Solution: $\exists x\left(F(x) \wedge T(x)\right)$

Some Fun with Translating from English into Logical Expressions 5

U = {fleegles, snurds, thingamabobs}

F(x): *x* is a fleegle.

S(x): *x* is a snurd.

T(x): *x* is a thingamabob.

"No snurd is a thingamabob."

Solution: $\neg\exists x(S(x) \wedge T(x))$ What is this equivalent to?

Solution: $\forall x(\neg S(x) \vee \neg T(x))$

Some Fun with Translating from English into Logical Expressions 6

U = {fleegles, snurds, thingamabobs}

F(x): x is a fleegle.

S(x): *x* is a snurd.

T(x): *x* is a thingamabob.

"If any fleegle is a snurd then it is also a thingamabob."

Solution: $\forall x\left(\left(F(x) \wedge S(x)\right) \rightarrow T(x)\right)$

System Specification Example

Predicate logic is used for specifying properties that systems must satisfy.

For example, translate into predicate logic:

- "Every mail message larger than one megabyte will be compressed."
- "If a user is active, at least one network link will be available."

Decide on predicates and domains (left implicit here) for the variables:

- Let $L(m, y)$ be "Mail message *m* is larger than *y* megabytes."
- Let $C(m)$ denote "Mail message *m* will be compressed."
- Let $A(u)$ represent "User *u* is active."
- Let $S(n, x)$ represent "Network link *n* is state *x*.

Now we have: $\forall m\left(L(m,1) \to C(m)\right)$

$$\exists u\, A(u) \to \exists n\, S(n,\, available)$$

Lewis Carroll Example

Charles Lutwidge Dodgson (AKA Lewis Caroll) (1832-1898)

The first two are called *premises* and the third is called the *conclusion*.

1. "All lions are fierce."
2. "Some lions do not drink coffee."
3. "Some fierce creatures do not drink coffee."

Here is one way to translate these statements to predicate logic. Let P(x), Q(x), and R(x) be the propositional functions "x is a lion," "x is fierce," and "x drinks coffee," respectively.

1. $\forall x\ (P(x) \rightarrow Q(x))$
2. $\exists x\ (P(x) \wedge \neg R(x))$
3. $\exists x\ (Q(x) \wedge \neg R(x))$

Later we will see how to prove that the conclusion follows from the premises.

Nested Quantifiers

Section 1.4

Section Summary 2

Nested Quantifiers.

Order of Quantifiers.

Translating from Nested Quantifiers into English.

Translating Mathematical Statements into Statements involving Nested Quantifiers.

Translated English Sentences into Logical Expressions.

Negating Nested Quantifiers.

Nested Quantifiers

Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

Example: "Every real number has an inverse" is

$$\forall x \exists y (x + y = 0)$$

where the domains of x and y are the real numbers.

We can also think of nested propositional functions:

$\forall x \exists y (x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is
$\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Thinking of Nested Quantification

Nested Loops

To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x :

- At each step, loop through the values for y.
- If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.

  $\forall x \forall y P(x,y)$ is true if the outer loop ends after stepping through each x.

To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:

- At each step, loop through the values for y.
- The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
- If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

  $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.

If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.
2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x\ \exists y Q(x,y)$ is true, but $\exists y\ \forall x Q(x,y)$ is false.

Questions on Order of Quantifiers 1

Example 1: Let *U* be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer : False

2. $\forall x \exists y P(x,y)$

   Answer : True

3. $\exists x \forall y P(x,y)$

   Answer : True

4. $\exists x \exists y P(x,y)$

   Answer : True

Questions on Order of Quantifiers 2

Example 2: Let *U* be the real numbers,

Define *P(x,y)* : *x* / *y* = 1

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   $\textbf{Answer:}$ False

2. $\forall x \exists y P(x,y)$

   $\textbf{Answer:}$ False

3. $\exists x \forall y P(x,y)$

   $\textbf{Answer:}$ False

4. $\exists x \exists y P(x,y)$

   $\textbf{Answer:}$ True

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair $x,\ y$ for which $P(x,y)$ is false.
$\forall x \forall y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair $x,\ y$ for which $P(x,y)$ is true.	P(x,y) is false for every pair x,y.

Translating Nested Quantifiers into English

Example 1: Translate the statement

$$\forall x\left(C(X) \vee \exists y\left(C(y) \wedge F(X,y)\right)\right)$$

where C(x) is "*x* has a computer," and *F*(*x*,*y*) is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x \forall y \forall z\left(\left(F(x,y) \wedge F(x,z) \wedge (y \neq z)\right) \rightarrow \neg F(y,z)\right)$$

Solution: There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:
   "For every two integers, if these integers are both positive, then the sum of these integers is positive."
2. Introduce the variables *x* and *y*, and specify the domain, to obtain:
   "For all positive integers *x* and *y*, *x* + *y* is positive."
3. The result is:

$$\forall x \forall y \left(\left(x > 0 \right) \wedge \left(y > 0 \right) \right) \rightarrow \left(x + y > 0 \right)$$

where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let *P(w,f)* be "*w* has taken *f* " and *Q*(*f,a*) be "*f* is a flight on *a* ."
2. The domain of *w* is all women, the domain of *f* is all flights, and the domain of *a* is all airlines.
3. Then the statement can be expressed as:

$$\exists w \forall a \exists f \left(P\left(w,f\right) \wedge Q\left(f,a\right)\right)$$

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x \forall y (B(\mathrm{x,y}) \rightarrow S(\mathrm{x,y}))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x \forall y (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x \exists y\ L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y \forall x\ L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x \exists y\ L(x,y)$

Example 6: "Everyone loves himself"

Solution: $\forall x\ L(x,x)$

Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:

$$\exists w \forall a \exists f \left(P(w,f) \wedge Q(f,a) \right)$$

Part 1: Use quantifiers to express the statement that “There does not exist a woman who has taken a flight on every airline in the world.”

Solution: $\neg \exists w \forall a \exists f \left(P(w,f) \wedge Q(f,a) \right)$

Part 2: Now use De Morgan’s Laws to move the negation as far inwards as possible.

Solution:a

1. $\neg \exists w \forall a \exists f \left(P(w,f) \wedge Q(f,a) \right)$
2. $\forall w \neg \forall a \exists f \left(P(w,f) \wedge Q(f,a) \right)$ by De Morgan’s for $\exists$
3. $\forall w \exists a \neg \exists f \left(P(w,f) \wedge Q(f,a) \right)$ by De Morgan’s for $\forall$
4. $\forall w \exists a \forall f \neg \left(P(w,f) \wedge Q(f,a) \right)$ by De Morgan’s for $\exists$
5. $\forall w \exists a \forall f \left(\neg P(w,f) \vee \neg Q(f,a) \right)$ by De Morgan’s for $\wedge$.

Part 3: Can you translate the result back into English?

Solution:

“For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline”